Optimal Control of Assemble-to-Order Systems

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Abstract: We consider the optimal control of an assemble-to-order (ATO) system with \( m \) components, a single end-product, and \( n \) customer classes. Demand from each class occurs continuously over time according to a Poisson process. Components are produced in separate production facilities, each with a finite production rate and exponentially distributed production times. Components can be stocked ahead of demand but incur a holding cost. When an order arises, it can be satisfied only if all \( m \) components are available in stock. Otherwise, it must either be backordered or rejected if backorders are not allowed. A backordered or rejected demand incurs a shortage cost (either a backorder or a lost sales cost), which may vary by demand class. At any time, the system manager must decide which components to produce and, if there is on-hand inventory, which orders from those pending to satisfy. We formulate the problem as a Markov decision process and characterize the structure of the optimal policy. We show that the optimal production policy for each component is a state-dependent base-stock policy, where the base-stock level for each component is non-decreasing in the inventory level of other components. We show that the optimal inventory allocation is a state-dependent multi-level rationing policy where the component rationing level for each class is non-decreasing in the inventory level of other components. Using numerical results, we compare the performance of the optimal policy to simple heuristics. For cases where it is possible to compare the heuristics to the optimal policy, we find the heuristics to perform well.

Keywords: Assemble-to-order (ATO) systems, production and inventory control, Markov decision processes, make-to-stock queues
1 Introduction

Assemble-to-order (ATO) production has become a popular strategy for manufacturing firms that seek to be both responsive and cost efficient. ATO production enables a firm to shorten its response time to its customers by staging inventory of components ahead of demand while postponing the final assembly until demand is realized. This strategy is particularly valuable when component supply leadtimes are long or the supply processes are capacitated. Furthermore, by pooling component inventories, ATO can reduce the costs of offering higher product variety, which can be useful when demand for individual end-products is variable. In addition to manufacturing, features of ATO systems are present in several other settings where demand is either correlated across several items (e.g., demand for computers and printers) or orders for multiple items from the same customer must be fulfilled simultaneously (e.g., order fulfillment at e-retailers and mail order catalogs).

In spite of their prevalence in practice, ATO systems are notoriously difficult to analyze and manage. ATO systems have been primarily analyzed via approximations and controlled via heuristics. With few exceptions, the optimal control policy remains unknown for most ATO systems (Song and Zipkin, 2003) (Wang, 2002) (Gallien and Wein, 2001). The difficulty appears due to several factors including (a) demands for the different components being correlated, (b) supply leadtimes for different components being different, and (c) order fulfillment being dependent on the availability of multiple components. This difficulty is compounded in systems with multiple products and multiple customer classes where inventory must be allocated among competing orders with varying priorities.

In this paper, we describe the optimal control policy for a system consisting of $m$ components, $n$ customer classes, and a single end-product (or multiple products requiring the same $m$ components). Demand from each class occurs continuously over time according to a Poisson process. Components are produced on separate production facilities with finite production rates and exponentially distributed production times. Components can be produced ahead of demand and held in stock, but incur a holding cost that varies by component. In order to fulfill a customer order all $m$ components must be available. Otherwise, the order is backordered or lost if backorders are not allowed. An order that is not immediately fulfilled from stock incurs a shortage cost (a backorder cost or a lost sales cost) that can vary from class to class. At any time the system manager must determine whether or not to produce a particular component and, if there is on-hand inventory, whether or not to allocate it to any of the pending orders.
We formulate the problem as a Markov decision process (MDP) and characterize the structure of the optimal policy. We show that the optimal production policy for each component is a state-dependent base-stock policy, where the state of the system is specified by the vector of component inventory levels. We show that the base-stock level for each component is non-decreasing in the inventory level of other components with a unit increase in the inventory of any component leading to at most one unit increase in the base-stock level of other components. We show that the optimal inventory allocation policy consists of a multi-level rationing policy. An order from a customer class is satisfied only if the current inventory level at each component is above a certain rationing level. The rationing levels for each class at each component are state-dependent and non-decreasing in the inventory levels of other components.

Using numerical results, we compare the performance of the optimal policy with the performance of two simple heuristics. The first heuristic consists of using a stationary base-stock level for each component. The second heuristic monitors the difference in the inventory level between each pair of components and stops the production of a component if this difference is above a target threshold. For both heuristics, inventory allocation is carried out via stationary rationing levels. For cases where it is possible to compare the heuristics to the optimal policy, we found the heuristics to perform surprisingly well for a wide range of parameter values.

The rest of this paper is organized as follows. In section 2, we offer a brief review of related literature. In section 3, we present our model for systems with a single demand class. In section 4, we extend our analysis to systems with multiple demand classes. In both sections 3 and 4, we also present numerical results and comparisons with heuristics. In section 5, we offer a summary and few concluding comments.

2 Literature Review

The literature on ATO can be broadly classified into two categories, one dealing with systems with periodic review and one with systems with continuous review (see Song and Zipkin (2003) for a comprehensive review). In both streams of literature the focus has been on performance evaluation of base-stock policies with stationary and independent base-stock levels for different components. In the special case of a system with a single product and deterministic supply leadtimes, Rosling (1989) shows that such policies are indeed optimal. However, for most other systems, as noted by Hausman et al.
(1998), Gallien and Wein (2001), Wang (2002), Song and Zipkin (2003) and many others, the structure of the optimal policy is not known.

Examples of papers dealing with ATO systems with periodic review include Hausmann et al. (1998), de Kok and Visschers (1999), and Cheng et al. (2002). Hausman et al. (1998) consider a system with multiple products and assume that a base-stock policy is used to replenish inventory. They assume that, in each period, inventory is allocated on a first-come, first served basis among orders currently backlogged. They evaluate the performance of the system via a bound on the probability of filling orders within a time window. de Kok and Visschers (1999) use a modified base-stock policy that adapts the approach in Rosling (1989) to systems with multiple products. Under a particular allocation policy, they show that the system reduces to a multi-level distribution system. Cheng et al. (2002) consider a system with multiple products and multiple components. Assuming component inventory is managed using a base-stock policy, they develop an optimization model with multiple constraints corresponding to service levels offered to different market segments. Other related models can be found in Gerchak and Henig (1989), Zhang (1997), Agrawal and Cohen (2001), Frank et al. (2004) and the references therein.

The literature on ATO systems with continuous review can be itself broadly classified into two areas based on the underlying component supply process: (1) systems with exogenous and load-independent leadtimes and (2) systems with endogenous and load-dependent leadtimes. In the first case, inventory replenishment leadtimes are assumed to be independent of the number of outstanding orders (pure inventory systems). In the second case, replenishment leadtimes are affected by the number of outstanding orders due to limitations in production capacity (integrated production-inventory systems). For both cases, component inventory is typically assumed to be managed using independent base-stock policies with stationary base-stock levels.

ATO systems with exogenous supply leadtimes (and stationary base-stock control) can be viewed as a set of infinite server queues -- i.e., $G/G/\infty$ queues -- with correlated arrivals. Song (1998) considers a system with Poisson demand and deterministic leadtimes modeled as $M/D/\infty$ queues. She develops an efficient algorithm for computing order fill rates. Song et al. (1999) extend this analysis to systems with exponential leadtimes; Song and Yao (2002) evaluate several performance bounds which they use to examine the tradeoff between inventory and service level. Gallien and Wein (1999) consider a system with a single product and i.i.d. leadtimes modeled as $M/G/\infty$ queues. They evaluate a policy consisting of
a fixed and common base-stock level for all components and a component-dependent order leadtimes, such that a replenishment order for each component is delayed by a component-dependent leadtime.

ATO systems with load-dependent supply leadtimes and stationary base-stock policies can also be viewed as queues with correlated arrivals but with a finite number of servers. Song et al. (1999) consider a system where each component is produced on a separate production facility with a single server and exponentially distributed production times. She assumes that there is a limit on the number $N$ of outstanding orders with each facility, so that each facility can be modeled as an $M/M/1/N$ queues. She shows that exact performance measures can be obtained using the matrix geometric method. Dayanik et al. (2003) develop bounds on the fill rate for systems that can be modeled as correlated $M/M/1/N$ queues. Glasserman and Wang (1998) consider systems that can be modeled as $M^X/G/1$ and $G^X/G/1$ queues and obtain an asymptotic relationship between base-stock levels and target delivery leadtimes. Based on an approximate formulation, they obtain a closed form solution for component base-stock levels. Wang (2001) considers the problem of minimizing inventory cost subject to a fill rate requirement. Plambeck and Ward (2004) study an ATO system where component production can be expedited at additional cost and demand is price-dependent. Under heavy traffic conditions, they develop a control policy for sequencing orders and expediting components that is optimal under heavy traffic conditions. Finally, there is a rich literature from queueing theory on the general analysis of systems with correlated arrivals. We refer the reader to Xu (2001) for a recent review.

The literature dealing with production and inventory control with multiple demand classes is relatively limited. Ha (1997a) considers a make-to-stock queue with a single component (no assembly). For a system with two customer classes and lost sales, he shows that the optimal policy is of the threshold type, where orders from the lower priority class are fulfilled as long as inventory is above a certain threshold level. Ha (1997b) extends these results to systems with backordering and de Véricourt et al. (2002) generalize them to systems with $N$ classes. Benjaafar et al. (2004) consider inventory rationing in a system with multiple products and multiple production facilities.

Our paper appears to be the first to consider the optimal control of an ATO system with continuous review and stochastic leadtimes. It also appears to be the first to characterize the structure of the optimal policy in a system with multiple demand classes. Furthermore, the results from the paper offer some numerical evidence regarding the performance of stationary base-stock policies and policies with stationary rationing levels.
3 Systems with a Single Demand Class

In this section, we consider systems with a single demand class. We first treat the case of systems with lost sales and then the one with backordering. In both cases, we first formulate the problem as an MDP and then characterize the structure of the optimal policy. We also present numerical results comparing the performance of the optimal policy to the performance of heuristics.

3.1 The Case of Lost Sales

We consider a system consisting of a single end-product assembled from $m$ components. The components are produced on $m$ separate production facilities and when finished are placed in stock. When demand for the end-product arises, the product is immediately assembled if all $m$ components are available otherwise the demand is considered lost (or must be expedited through other means including overtime or outsourcing to a third party). A demand that cannot be immediately fulfilled from stock incurs a *lost sales* cost $c$ per unit. Demand for the end-product takes place continuously over time according to a Poisson process with rate $\lambda$. The time to produce component $k$ is exponentially distributed with rate $\mu_k$, for $k=1,\ldots,m$. The state of the system at time $t$ can be described by the vector $X(t) = (X_1(t),\ldots,X_m(t))$, where $X_k(t)$ is a non-negative integer denoting the on-hand inventory for component $k$ at time $t$. We let $h_k(X_k(t))$ denote the inventory holding cost incurred at time $t$, where $h_k$ is an increasing convex function.

At any time $t$ ($t \geq 0$), the system manager may choose either to produce any of the components or not. If a component is not currently being produced, this means deciding whether or not to initiate its production. If the component is currently being produced, this means deciding whether or not to interrupt its production (as we show in theorem 1, it turns out that it is never optimal to interrupt production of a component once it is initiated). Because both order inter-arrival times and production times are exponentially distributed, the system is memoryless and decision epochs can be restricted to only times when the state changes (i.e., the arrival of a new order or the completion of a component). Also, because of the memoryless property, we can restrict our attention to the class of Markovian policies for which actions taken at a particular decision epoch depend only on the current state of the system.

Let $v^\pi(x)$ denote the expected discounted cost obtained under a policy $\pi$ and a starting state $x=(x_1,\ldots,x_m)$. Then $v^\pi(x)$ is given by

$$v^\pi(x) = E_x\left[\int_0^\infty e^{-at}h(X(t)) \, dt + \int_0^\infty e^{-at}c \, dN(t)\right], \quad (1)$$

\[\]
where \( \alpha > 0 \) is the discount factor and \( N(t) \) is the number of orders that have not been satisfied up to time \( t \). Our objective is to choose a policy \( \pi^* \) that minimizes the expected discounted cost. Following Lippman (1975), we introduce the uniform discount rate \( \beta = \lambda + \sum_{k=1}^{m} \mu_k \) and, without loss of generality, rescale time so that \( \alpha + \beta = 1 \). Then, the optimal cost function \( v^* \equiv v_{\pi^*} \) can be shown to satisfy the following optimality equation for any starting state \( x \):

\[
v^*(x) = h(x) + \lambda T_0 v^*(x) + \sum_{k=1}^{m} \mu_k T_k v^*(x).
\]

The operators \( T_k (k = 0, \ldots, m) \) are defined as follows:

\[
T_0 v(x) = \begin{cases} 
v(x) + c & \text{if } \prod_{k=1}^{m} x_k = 0, \\
v(x - e) & \text{otherwise}, \end{cases}
\]

and

\[
T_k v(x) = \min\{v(x + e_k), v(x)\},
\]

where \( e_k \) is the \( k \)-th unit vector of dimension \( m \) and \( e = \sum_{k=1}^{m} e_k = (1, 1, \ldots, 1) \) is an \( m \)-dimensional vector of ones. The operator \( T_0 \) corresponds to the decision of accepting or rejecting an order and \( T_k \) \( (k = 1, \ldots, m) \) to the decision of either producing component \( k \) or not. Note that whenever \( v(x) > v(x + e_k) \), it is optimal to produce component \( k \).

In the following theorem we characterize the structure of the optimal policy. The proof of this and all subsequent results are included in the Appendix.

**Theorem 1:** The optimal control policy consists of a state-dependent base-stock policy with a vector of state-dependent base-stock levels \( s^* = (s_1^*(x_{1-}), \ldots, s_m^*(x_{m-})) \) such that it is optimal to produce component \( k \) if \( x_k < s_k^*(x_{-k}) \) and not to produce it otherwise, where \( x_k \) is the on-hand inventory level of component \( k \) and \( x_{-k} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \) is an \( (m-1) \) dimensional vector consisting of the on-hand inventory levels for components \( j \neq k \). The optimal policy has the following additional properties.

1. The optimal base-stock level \( s_k^*(x_{-k}) \) for component \( k \) is non-decreasing in the on-hand inventory levels of other components. That is, \( s_k^*(x_{-k}) \) is non-decreasing in each of the variables \( x_j, j \neq k \).
2. A unit increase in the inventory level of component \( j \) leads to at most a unit increase in the optimal base-stock level for component \( k \) \( (k \neq j) \). In other words, \( s_k^*(x_{-k} + e_j) \leq s_k^*(x_{-k}) + 1 \).
3. It is never optimal to interrupt the production of a component once it is initiated.

Theorem 1 states that each component can be managed using a base-stock policy. However, the base-stock level for each component depends on the current inventory level of other components. In particular,
an increase in the inventory level of a component leads either to an increase in the base-stock level of other components or leaves these base-stock levels unchanged. Interestingly, one unit increase in the inventory level of one component leads to at most one unit increase in the base-stock level of the other components (i.e., $s_i^*$, as a function of state variable $x_j$, is bounded by a linear function with unit slope). Furthermore, it is never optimal to reject an order if it can be immediately fulfilled from on-hand inventory (as we show in section 5, this property ceases to hold for systems with multiple demand classes) and it is always optimal to let a component complete its production once it has started.

In Figure 1, we illustrate the optimal policy for a system with two components. As we can see, the base-stock level functions $s_1^*(x_2)$ and $s_2^*(x_1)$ divide the state space into four regions: region 1 where both components are produced, region 2 where component 1 is produced but not 2, region 3 where component 2 is produced but not 1, and region 4 where neither component is produced. It is important to note that other than region 1, the other three regions are transient under the optimal policy.

![Figure 1: The structure of the optimal policy](image)

3.2 The Case of Backorders

In this section, we consider systems where orders that cannot be fulfilled immediately from stock are backordered and not lost. In this case, the state of the system at time $t$ is described by the vector.
\[ Y(t) = (Y_1(t), \ldots, Y_m(t)) \] where \( Y_k(t) \) is the inventory position for component \( k \) at time \( t \), such that inventory level \( X_k(t) \) is given by \( X_k(t) = [Y_k(t)]^+ \) and backorder level \( B(t) \) is given by \( B(t) = \max_k \{[Y_k(t)]^- \} \), where \([Y_k(t)]^+ = \max(0, Y_k(t))\) and \([Y_k(t)]^- = \max(0, -Y_k(t))\). Let \( bB(t) \) denote the backorder cost at time \( t \), where \( b > 0 \). Then, the total cost \( z(Y(t)) \) incurred at time \( t \) can be written as

\[
z(Y(t)) = bB(t) + \sum_{k=1}^m h_k(X_k(t)),
\]

or equivalently as

\[
z(Y(t)) = b(\max_k \{[Y_k(t)]^- \}) + \sum_{k=1}^m h_k([Y_k(t)]^-).
\]

At every time \( t \), a control policy \( \pi \) specifies whether or not any of the components should be produced. Similar to the case with lost sales, we can restrict our attention to the class of Markovian policies and we can limit decision epochs to only times when the state changes. Let \( v^\pi(y) \) denote the expected discounted cost obtained under a policy \( \pi \) and a starting state \( y = (y_1, \ldots, y_m) \). Then \( v^\pi(y) \) is given by

\[
v^\pi(y) = E_y^{\pi} \left[ \int_0^\infty e^{-\alpha t} z(Y(t))dt \right].
\]

Our objective is to choose a policy \( \pi^* \) which minimizes the expected discounted cost. After rate uniformization and time rescaling, the optimal cost function \( v^* \) can be shown to satisfy the following optimality equation for any starting state \( y \):

\[
v^*(y) = z(y) + \lambda v^*(y - e) + \sum_{k=1}^m \mu_k T_k v^*(y).
\]

The operator \( T_k \) \((k=1, \ldots, m)\) is defined as:

\[
T_k v(y) = \min \{ v(y + e_k), v(y) \},
\]

and corresponds to the decision of whether or not component \( k \) should be produced. This decision is made each time a new order arrives or any component completes production.

Theorem 2 characterizes the structure of the optimal policy which retains all the properties observed in systems with lost sales, with the additional property that it is optimal to produce a component whenever its net inventory is negative.

**Theorem 2:** The optimal control policy consists of a state-dependent base-stock policy with a vector of state-dependent base-stock levels \( s^* = (s_1^*(y_1), \ldots, s_m^*(y_m)) \) such that it is optimal to produce component \( k \) if \( y_k < s_k^*(y) \) and not to produce it otherwise, where \( y_k \) is the net inventory of component \( k \) and \( y_k = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m) \) is an \((m-1)\) dimensional vector consisting of net inventory levels for all components \( j \neq k \). The optimal policy has the following additional properties.
(1) The optimal base-stock level \( s^*_k(y_{-k}) \) for component \( k \) is non-decreasing in the net inventory levels of other components. That is, \( s^*_k(y_{-k}) \) is non-decreasing in each of the variables \( y_j, j \neq k \).

(2) A unit increase in the net inventory of component \( j \) leads to at most a unit increase in the optimal base-stock level for component \( k (k \neq j) \). In other words, \( s^*_k(y_{-k} + e_j) \leq s^*_k(y_{-k}) + 1 \).

(3) It is always optimal to produce a component if its net inventory is negative, i.e., \( s_k(y_{-k}) \geq 0 \).

(4) It is never optimal to interrupt the production of a component once it is initiated.

### 3.3 Heuristic Policies

In this section, we consider two simple heuristic policies. In section 3.4, we compare their performance to the optimal policy. The first heuristic consists of controlling the production of components using stationary and independent base-stock levels. We refer to this heuristic as the independent base-stock policy (IBP). The policy is specified in terms of a vector of base-stock levels \( s = (s_1, ..., s_m) \).

For systems with lost sales, the expected discounted cost for the IBP heuristic can be computed (for a given vector \( s \) of base-stock levels) via the following dynamic programming equation:

\[
v^{IBP}_*(x) = h(x) + \lambda T_0 v^{IBP}_*(x) + \sum_{k=1}^{m} \mu_k T_{k}^{IBP} v^{IBP}_*(x),
\]

where

\[
T_0 v^{IBP}_*(x) = \begin{cases} v^{IBP}_*(x) + c & \text{if } \prod_{k=1}^{m} x_k = 0, \\ v^{IBP}_*(x - e) & \text{otherwise,} \end{cases}
\]

and

\[
T_k^{IBP} v^{IBP}_*(x) = \begin{cases} v^{IBP}_*(x + e_k) & \text{if } x_k < s_k, \\ v^{IBP}_*(x) & \text{otherwise.} \end{cases}
\]

The vector of optimal base-stock levels \( s^{IBP} = (s_1^{IBP}, ..., s_m^{IBP}) \) can be obtained using a grid search for a sufficiently large range of base-stock values.

For systems with backorders, the expected discounted cost can be similarly computed via the following dynamic programming equation:

\[
v^{IBP}_*(y) = z(y) + \lambda v^{IBP}_*(y - e) + \sum_{k=1}^{m} \mu_k T_{k}^{IBP} v^{IBP}_*(y),
\]

where

\[
T_k^{IBP} v(y) = \begin{cases} v^{IBP}_*(y + e_k) & \text{if } y_k < s_k, \\ v^{IBP}_*(y) & \text{otherwise.} \end{cases}
\]

The second heuristic policy, to which we refer as the coordinated base-stock policy (CBP), employs a vector of base-stock levels \( s = (s_1, ..., s_m) \) and a coordination parameter \( r \). The CBP heuristic works as
follows. When net inventory of component \( k \) is less than the base-stock level \( s_k \), production is not initiated unless the difference between the net inventory level of component \( k \) and the smallest net inventory level among the other components is less than an amount \( r \). For systems with lost sales, the expected discounted cost for the CBP heuristic can be computed via the following dynamic programming equation:

\[
v^{\text{CBP}}(x) = h(x) + \lambda T_0 v^{\text{CBP}}(x) + \sum_{k=1}^{m} H_k T_k^{\text{CBP}} v^{\text{CBP}}(x),
\]

where

\[
T_k^{\text{CBP}} v(x) = \begin{cases} 
  v(x + e_k) & \text{if } x_k < \min_{j \neq k \{s_j, x_j + r\}}, \\
  v(x) & \text{otherwise.}
\end{cases}
\]

Note that the IBP heuristic is a special case of the CBP heuristic and corresponds to the case where \( r \geq \max(s_1, \ldots, s_m) \). Hence, the performance of the IBP heuristic provides an upper bound on that of the CBP heuristic.

A similar equation to (15) can be written for systems with backorders, where the operators \( T_k \) are modified as follows:

\[
T_k^{\text{CBP}} v(y) = \begin{cases} 
  v(y + e_k) & \text{if } y_k < 0 \\
  v(y + e_k) & \text{if } 0 \leq y_k < \min_{j \neq k \{s_j, x_j + r\}} \\
  v(y) & \text{otherwise.}
\end{cases}
\]

This means that a component is produced whenever its net inventory drops below zero regardless of the inventory status of other components. Note that the CBP heuristic can be further refined by letting the coordination parameter be component-dependent. This leads to a coordination vector \( r=(r_1, \bar{r}, r_m) \), where \( r_k \) is the coordination parameter used by component \( k \).

### 3.4 Numerical Results

To test the performance of the optimal policy against the two heuristics, we carried out a series of numerical experiments for a system with two components and for a wide range of parameter values. For brevity, we present results only for the lost sales case. The results are qualitatively the same for systems with backorders. For each problem instance, the expected average cost is obtained by solving the corresponding dynamic program using the value iteration method. The state space is truncated at \( \{0,n_1\} \times \{0,n_2\} \), where \( n_1 \) and \( n_2 \) are gradually increased until the expected cost is no longer sensitive to the truncation level. The value iteration algorithm is terminated once five-digit accuracy is obtained. The holding costs are linear with \( h_1 \geq 0 \) and \( h_2 \geq 0 \) being the unit holding costs per unit time for components 1 and 2 respectively. Note that we present results for the average cost instead of the expected discounted
cost. We do so because it is computationally more convenient and because the results are independent of
the starting state and the discount factor. The structure of the optimal policy can be shown to remain the
same under the average cost criteria (see for example Weber and Stidham (1987) for discussion and
references).

Representative results from our numerical experiments are shown in Figures 2-5 and Table 1. Figures
2-5 show the average cost for both the optimal policy and the CBP heuristic and the percentage difference
between the two, which corresponds to the relative cost reduction (CR) from using the optimal policy.
Only results for the CBP heuristic are shown in Figures 2-5. Results for the IBP closely parallel those of
the CBP heuristic. The performance of the CBP and IBP heuristics are compared in Table 1. Based on the
figures, the following observations can be made.

1. Using the optimal policy instead of the heuristics can lead to significant percentage reductions in
average cost, over 150% in some cases (see Figure 2).

2. As shown in Figure 2, the percentage cost reduction from using the optimal policy is most significant
when the lost sales cost \( c \) is small. This makes intuitive sense since when \( c \) is high the base-stock
levels for both components are high, whether we use the optimal policy or one of the heuristics,
which diminishes the value from coordination among components.

3. As illustrated in Figure 3, the percentage cost reduction increases with increases in the ratio \( \mu_1/\mu_2 \) (for
fixed \( \mu_1 + \mu_2 \)). This is not surprising since one would expect coordination between the two
components to be more important when there are large differences in their production rates.

4. As shown in Figure 4, the percentage cost reduction tends to decrease when the demand rate \( \lambda \)
increases. This is due to the fact that we tend to produce both components most of the time when \( \lambda \) is
high, regardless of which policy is used.

5. As seen in Figure 5, the percentage cost reduction tends to decrease in the ratio \( h_1/h_2 \) (for fixed \( h_1+h_2 \)).
This can be explained as follows. When the ratio \( h_1/h_2 \) is high, the base-stock levels for component 2
are high. The value of coordination among components diminishes since inventory from component 2
is generally available.

6. An increase \( h_1/h_2 \) can lead to a decrease in the average costs of both the optimal policy and the
heuristics when \( \mu_1 \) is much larger than \( \mu_2 \). Since an increase in \( h_1 \) leads to a decrease in \( h_2 \) (for fixed
\( h_1+h_2 \)), the base-stock level of component 2 increases in \( h_2 \) which, in turn, reduces shortages due to
the unavailability of component 2. The increase in \( h_1 \) does increase the cost of holding component 1.
Figure 2 - The effect of lost sales cost
\[\mu_1 + \mu_2 = 2.0, \frac{\mu_1}{\mu_2} = 25.0, \rho_2 = 0.9, \lambda = \rho_2 \mu_2, h_1 = h_2 = 2.0\]

Figure 3 - The effect of production rate asymmetry
\[\mu_1 + \mu_2 = 2.0, \rho_2 = 0.9, \lambda = \rho_2 \mu_2, h_1 = h_2 = 2.0, c = 200\]
Figure 4 - The effect of demand rate ($\lambda$)
($\mu_1 + \mu_2 = 2.0$, $\mu_1/\mu_2 = 25.0$, $h_1 = h_2 = 2$, $c = 200$)

Figure 5 - The effect of the ratio of holding costs
($\mu_1 + \mu_2 = 2.0$, $\mu_1/\mu_2 = 25.0$, $\rho_2 = 0.9$, $\lambda = \rho_2 \mu_2$, $h_1 + h_2 = 4.0$, $c = 200$)
However, the impact of this increase is less significant since the production rate of component 1 is high, mitigating the need for high inventory of component 1. This is of course not possible when the production rate of component 1 is low relative to that component 2, and in that case we do observe the opposite effect of average costs increasing in $h_1/h_2$ (for brevity the results are not shown).

7. Although the percentage cost difference between the optimal policy and the heuristics can be significant, the absolute difference between the average costs is surprisingly small. The largest percentage difference is observed when the average costs are small. This tends to occur in ranges of parameter values that are unlikely to occur in practice (e.g., very low demand, very low backorder costs, or very large differences between component production rates).

8. As shown in Table 1, the difference in cost, both absolute and relative, between the two heuristics is small for most of the cases generated. The difference between the two policies quickly diminishes as the ratio $\mu_1/\mu_2$ increases.

The above results appear to suggest that the heuristics could serve as effective alternatives to the optimal policy. In particular, the numerical results seem to support the belief (at least for a system with two components) that the IBP heuristic is indeed effective. It is not clear if the results would continue to hold for systems with a large number of components (we do observe similar results for systems with three components). Unfortunately, obtaining numerical results for systems with a large number of components is difficult due to the exponential growth in the dimensions of the state space.

Table 1: Percentage cost difference between IBP and CBP heuristics
($\mu_1 + \mu_2 = 2.0$, $\rho_2 = 0.9$, $\lambda = \rho_2 \mu_2$, $h_1 = h_2 = 2.0$)

<table>
<thead>
<tr>
<th>$\mu_1/\mu_2$</th>
<th>c=50</th>
<th>c=100</th>
<th>c=200</th>
<th>c=300</th>
<th>c=400</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1.091</td>
<td>0.895</td>
<td>0.933</td>
<td>0.699</td>
<td>0.727</td>
</tr>
<tr>
<td>1.5</td>
<td>0.884</td>
<td>0.154</td>
<td>0.008</td>
<td>0.005</td>
<td>0.012</td>
</tr>
<tr>
<td>2.0</td>
<td>0</td>
<td>0.000</td>
<td>0.004</td>
<td>0.000</td>
<td>0.001</td>
</tr>
<tr>
<td>2.5</td>
<td>0</td>
<td>0.047</td>
<td>0.000</td>
<td>0.005</td>
<td>0.000</td>
</tr>
<tr>
<td>3.0</td>
<td>0</td>
<td>0.033</td>
<td>0.000</td>
<td>0.000</td>
<td>0.002</td>
</tr>
<tr>
<td>3.5</td>
<td>0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>
4 Systems with Multiple Demand Classes

In this section, we consider systems where demand arises from \( n \) different customer classes. The classes differ in their demand rates and their shortage penalties. The fact that different classes have different shortage costs can lead to situations where it might be more beneficial not to satisfy demand from a class in order to reserve inventory for future demand from a more important class (i.e., one with a higher shortage cost). Hence in addition to determining whether or not to produce a component, an optimal policy must determine whether or not to satisfy an incoming order.

4.1 The Structure of the Optimal Policy

We restrict our treatment to systems with lost sales (although we do not undertake it in this paper, the structure of the optimal policy, based on numerical results, appear to remain the same for systems with backorders). Let \( \lambda_l \) represent the arrival rate for demand class \( l \) which occurs according to an independent Poisson process and let \( c_l \) denote the lost sales cost for class \( l \) (without loss generality, we assume \( c_1 \geq \ldots \geq c_n \)). As in systems with a single demand class, the state at time \( t \) can be described by the vector of component inventory \( X(t) = (X_1(t), \ldots, X_m(t)) \). At any time \( t \ (t \geq 0) \), a control policy \( \pi \) specifies whether or not each component should be produced. With each order arrival, the control policy also specifies, whenever there is available inventory, whether or not the order should be satisfied. As in systems with a single class, the objective is to minimize the sum of inventory holding and lost sales cost. The expected discounted cost \( v^\pi(x) \) obtained under a control policy \( \pi \) and starting state \( x=(x_1, \bar{O}, x_m) \), is then given by

\[
v^\pi(x) = E^\pi \left[ \int_0^\infty e^{-\alpha t} h(X(t)) \, dt + \sum_{l=1}^n \int_0^\infty e^{-\alpha t} c_l dN_l(t) \right], \tag{18}
\]

where \( N_l(t) \) represent the number of orders of type \( l \) that are not fulfilled from inventory up to time \( t \). A similar development to the single class case leads to the following optimality equation:

\[
v^*(x) = h(x) + \sum_{l=1}^n \lambda_l T_0^l v^*(x) + \sum_{k=1}^m \mu_k T_k v^*(x) \tag{19}
\]

where the operator \( T_0^l \) for \( l = 1, \bar{O}, n \) is defined as follows:

\[
T_0^l v(x) = \begin{cases} 
      v(x) + c_l & \text{if } \prod_{k=1}^n x_k = 0 \\
      \min \{ v(x-e), v(x) + c_l \} & \text{otherwise}, 
   \end{cases} \tag{20}
\]
and corresponds to the decision about whether to satisfy an incoming order of type \( l \) or not. The definition and interpretation of the operator \( T_k \) remains unchanged from the single demand class case.

**Theorem 3:** The optimal policy for each component \( k, k = 1, \ldots, m \), consists of a base-stock level \( s^*_k(x_{-k}) \) and a vector of rationing levels \( r^*_k(x_{-k}) = (r^*_{k,1}(x_{-k}), \ldots, r^*_{k,n}(x_{-k})) \) such that it is optimal to produce component \( k \) if \( x_k < s^*_k(x_{-k}) \) and not to produce it otherwise and to fulfill an incoming order from class \( l \), \( l = 1, \ldots, n \), if \( x_k \geq r^*_{k,l}(x_{-k}) \) for \( k = 1, \ldots, m \) and not to fulfill it otherwise. In addition to preserving all the properties observed in the single class case, the optimal policy has the following properties.

1. **(1)** The rationing level \( r^*_{k,l}(x_{-k}) \) for class \( l \) at component \( k \) is non-increasing in the on-hand inventory levels of other components. That is, \( r^*_{k,l}(x_{-k}) \) is non-increasing in each of the variables \( x_j, j \neq k \).
2. **(2)** The rationing levels are ordered as follows \( r^*_{k,n}(x_{-k}) \geq \ldots \geq r^*_{k,1}(x_{-k}) = 0 \).
3. **(3)** It is always optimal to satisfy a demand from class 1 if there is on-hand inventory.

Theorem 3 states that the production of each component is still governed by a base-stock policy, with a base-stock level that depends on the inventory level of all the other components. An increase in the inventory level of a component again leads to either an increase in the base-stock level of other components or leaves these base-stock levels unchanged. Furthermore, the theorem states that inventory allocation is controlled by a rationing policy that determines whether a demand from a particular class is satisfied or not. In particular, a demand from a class is satisfied only if the inventory of each component is above its rationing level for that demand class. Note that the rationing levels for a class are not necessarily identical across components. Moreover, an increase in the inventory level of one component could lead to a decrease in the rationing levels at other components.

The structure of the optimal policy is illustrated in Figures 7 and 8 for a system with three demand classes and two components. As we can see from Figure 7, the optimal rationing policy divides the state space into three regions: region 1 where demand from any class is satisfied, region 2 where demand from only class 2 and 3 is satisfied, and region 3 where only demand from class 1 is satisfied. The boundaries of these regions highlight the fact that the rationing level at one component can be affected in dramatic ways by the inventory level at other components. For example, an increase in the inventory of component 1 from 3 to 4 leads the rationing level for class 2 to drop from 12 to 3. As illustrated in Figure 8, the optimal production policy partitions the state space into four regions where in region 1 we produce both components, in regions 2 and 3 we produce one of the components but not the other, and in region 4 we produce none of the components. Similar to the single class case, only the region where we produce both
Figure 7 ñ The structure of the optimal rationing policy

Figure 8 ñ The structure of the optimal production policy
components is recurrent. The other regions are transient. The superposition of Figures 7 and 8 would partition the state space into several sub-regions, each corresponding to a combination of (a) satisfying one or more demand classes and (b) producing one or both components or not producing at all.

### 4.2 Heuristic Policies

In this section, we propose two heuristics that are counterparts to the IBP and CBP heuristics we considered for systems with a single class in section 3.3. The first one consists of controlling the production of components via a vector \( s = (s_1, \ldots, s_m) \) of stationary and independent base-stock levels and controlling acceptance/rejection of orders via a vector \( r_l = (r_{1,l}, \ldots, r_{m,l}) \) of stationary and independent rationing levels for each class \( l \), \( l = 1, \ldots, n \), such that an order from class \( l \) is fulfilled from on-hand inventory only if \( x_k \geq r_{k,l} \) for all \( k = 1, \ldots, m \). We refer to this policy as the independent base-stock with rationing (IBR) policy. The expected cost of the IBR policy can be obtained (for a given vector of base-stock levels \( s \) and a matrix of rationing levels \( r = [r_{k,l}] \)) via the following dynamic programming equation:

\[
v^{IBR}(x) = h(x) + \sum_{l=1}^{n} \lambda_l T_0^{l,IBR} v^{IBR}(x) + \sum_{k=1}^{m} \mu_k T_k^{IBR} v^{IBR}(x),
\]

where

\[
T_k^{IBR}(x) = \begin{cases} v(x + e_k) & \text{if } x_k < s_k, \\ v(x) & \text{otherwise}. \end{cases}
\]

\[
T_0^{l,IBR}(x) = \begin{cases} v(x - e) & \text{if } x \geq r_l, \\ v(x) & \text{otherwise}, \end{cases}
\]

The second heuristic is similar to IBR, except that we now introduce a coordination parameter \( r \), such that the production of a component \( k \) is not initiated unless the difference between the inventory level of component \( k \) and the smallest inventory level among the other components is less than \( r \). We refer to this policy as the coordinated base-stock with rationing (CBR) policy. Expected cost for the CBR policy can be obtained by solving the following dynamic programming equation:

\[
v^{CBR}(x) = h(x) + \sum_{l=1}^{n} \lambda_l T_0^{l,CBR} v^{CBR}(x) + \sum_{k=1}^{m} \mu_k T_k^{CBR} v^{CBR}(x),
\]

where

\[
T_k^{CBR}(x) = \begin{cases} v(x + e_k) & \text{if } x_k < \min_{j \neq k} \{s_j, x_j + r\}, \\ v(x) & \text{otherwise}, \end{cases}
\]

and
\[ T_{0}^{\text{CBR}}(v(x)) = \begin{cases} v(x - e) & \text{if } x \geq r, \\ v(x) & \text{otherwise.} \end{cases} \] (25)

4.3 Numerical Results

We carried out a series of numerical experiments first to examine the impact of differences in the shortage costs among different classes, second to assess the benefit of rationing inventory among different classes as indicated by the optimal policy, and third to evaluate the performance of the CBR and IBR heuristics.

We consider a system with two components and two demand classes, 1 and 2, with lost sales costs \( c_1 \) and \( c_2 \) respectively. We obtain the average cost for the system (1) under the optimal policy, (2) under a policy where orders are fulfilled on a first come first served (FCFS) regardless of their class, and (3) under the CBR and IBR heuristics. Under the FCFS policy, all demand classes are treated the same. Hence, the optimal average cost is obtained by aggregating the arrivals into a single class and solving the same dynamic program as in section 3.1.

Representative results from our experiments are shown in Figure 9 and Table 2. In Figure 9(a), we compare the average cost of the optimal policy to that of the FCFS policy as we vary the ratio \( c_1/c_2 \) by increasing \( c_1 \) and holding \( c_2 \) fixed. In Figure 9(b) we do the same except that we vary the ratio \( c_1/c_2 \) by increasing \( c_1 \) and decreasing \( c_2 \) while holding \( c_1 + c_2 \) constant. In Table 2, we compare the average cost of the optimal policy to the average cost of the two heuristics as we vary the ratio \( c_1/c_2 \) while holding \( c_1 + c_2 \) fixed. Based on the figures and the table, the following observations can be made.

1. As illustrated in Figure 9(a), the difference in average cost between the optimal policy and the FCFS policy increases dramatically as the cost of one of the demand classes increases. For large values of \( c_1/c_2 \), this difference can be significant. For example, when \( c_1/c_2 = 10 \), the average cost of the FCFS policy is more than triple that of the optimal policy. Although the average cost increases under both policies, the increase under the optimal policy is relatively modest even when the ratio \( c_1/c_2 \) is high while the increase under the FCFS is linear.

2. As shown in Figure 9(b), the difference in average cost between the optimal policy and the FCFS policy also increases as one demand class becomes more expensive and the other cheaper, with the differences becoming significant when \( c_1/c_2 \) is large. However, in contrast to the results of Figure 9(a), the average cost of the FCFS policy is unaffected by changes in \( c_1/c_2 \) while that of the optimal policy actually decreases.
3. The above results can be explained as follows. Under the FCFS policy, the probability of a rejected order being of class \( l \) is \( \frac{\lambda_i}{\sum \lambda_i} \) and the expected lost sales cost of a rejected order is \( \sum c_i \left( \frac{\lambda_i}{\sum \lambda_i} \right) \). Consequently, the total expected cost would increase linearly with increases in the lost sales cost of an individual class as long as the costs of the other classes are held fixed. This is the case for the system shown in Figure 9(a). However, the total expected cost would remain unchanged if an increase in the lost sales cost of one class is offset by a reduction in the cost of another class so that \( \sum c_i \left( \frac{\lambda_i}{\sum \lambda_i} \right) \) remains constant. This is true for the system shown in Figure 9(b) where the quantity \( \sum c_i \left( \frac{\lambda_i}{\sum \lambda_i} \right) = c_1 + c_2 \) is held fixed.

4. Increasing \( c_1 \) while decreasing \( c_2 \) allows the optimal policy to ration inventory more aggressively in favor of class 1. This reduces the probability that a class 1 demand is rejected. Of course, this may increase the probability that class 2 demand is rejected. However, since \( c_2 \) is decreasing with increases in \( c_1/c_2 \), the overall effect, as shown in Figure 9(b), is a decrease in total expected cost under the optimal policy.

5. As shown in Table 2, both the CBR and IBR heuristics perform well for a wide range of values of \( c_1/c_2 \). In all cases, the percentage difference between the heuristics and the optimal policy is less than 4 percent. The difference in cost between the two heuristics is small (less than 1 percent in all the cases we tested).

| \( c_1 + c_2 = 400 \), \( \mu_1 = 4/3 \), \( \mu_2 = 2/3 \), \( \lambda_1 = 0.7 \), \( \lambda_2 = 0.4 \), \( h_1 = 1 \), \( h_2 = 3.0 \) | \hline
| 0.5 | 29.958 | 29.958 | 30.411 | 30.411 |
| 1 | 29.958 | 32.499 | 27.088 | 27.088 |
| 2 | 23.564 | 33.770 | 24.366 | 24.366 |
| 3 | 20.305 | 35.041 | 20.692 | 20.692 |
| 5 | 17.200 | 36.196 | 17.757 | 17.757 |
| 10 | 16.010 | 36.629 | 16.550 | 16.550 |
| 15 | 15.381 | 36.856 | 15.914 | 15.914 |
5 Summary and Concluding Comments

In this paper, we considered the optimal control of an assemble-to-order system consisting of a single product, \( m \) components, and \( n \) customer classes. We formulated the problem as a Markov decision process and characterized the structure of the optimal control policy. In particular, we showed that the optimal production policy for each component is a state-dependent base-stock policy, where the base-stock level for each component is non-decreasing in the inventory level of other components. We showed that the optimal inventory allocation is a state-dependent multi-level rationing policy where the component rationing level for each class is non-decreasing in the inventory level of other components. Because obtaining the optimal base-stock levels and the optimal rationing levels can be computing-intensive when the number of components is large, we proposed simple heuristics with stationary parameters. The parameter values can be optimized using simulation or other performance evaluation methods. For cases where it is possible to compare the heuristics to the optimal policy, we found the heuristics to perform well.

The results presented in this paper provide a first step toward a better understanding of the structure of the optimal policy for assembly systems and for identifying useful heuristics. There are several avenues for future research. In particular, it would be of interest to consider systems with multiple products, where each product may require only a subset of the components, systems with variable order sizes where the order size distribution varies with product and demand class, and systems with multiple echelons where the assembly takes place over several stages. We expect the problem to become considerably more difficult with each additional feature and it is not clear if the optimal policy would continue to have a recognizable structure. However, it may still be possible to identify effective heuristics.

Acknowledgments: We would like to thank Yves Dallery, Jean-Philippe Gayon, Chung-Yee Lee, Francis de Véricourt, and Larry Zhou for many useful discussions and Dan Zhang for helpful comments on an earlier version of the paper.
Figure 9 (a) The benefit of inventory rationing

9(a): $c_2=200$, $\mu_1=\mu_2=1.0$, $\lambda_1=\lambda_2=0.7$, $h_1=h_2=2.0$

Figure 9 (b) The benefit of inventory rationing

9(b): $c_1+c_2=400$, $\mu_1=\mu_2=1.0$, $\lambda_1=\lambda_2=0.7$, $h_1=h_2=2.0$
References


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Ha, A. Y., iStock-Rationing Policy for a Make-to-Stock Production System with Two Priority Classes and Backordering,i Naval Research Logistics, 44, 457-472, 1997b.


Appendix

Proof of Theorem 1

First, we introduce the following difference operators

\[ \Delta_j v(x) = v(x + e_j) - v(x), \]

\[ \Delta_{i_1, i_2, \ldots, i_k} v(x) = v(x + e_{i_1} + e_{i_2} + \cdots + e_{i_k}) - v(x), \]

and combinations of such operators, including

\[ \Delta_{i, j} v(x) = \Delta_i \Delta_j v(x) = \Delta_j v(x + e_i) - \Delta_i v(x). \]

Note that the order in which the differences are taken does not matter i.e., \( \Delta_{i, j} v(x) = \Delta_{j, i} v(x) \). For notational convenience, we also define

\[ \Delta_{i} v(x) = \Delta_{1, 2, \ldots, m} v(x), \text{ and} \]

\[ \Delta_{e_i} v(x) = v(x + e - e_i) - v(x). \]

Next, let \( \mathcal{V} \) be the set of functions on \( \Box^{m} \) such that if \( v \in \mathcal{V} \) then for all \( i = 1, \ldots, m \):

A1: \( \Delta_{i} v(x) \geq 0 \), for all \( i \),

A2: \( \Delta_{i} v(x) \leq 0 \), for all \( i \) and \( j \neq i \),

A3: \( \Delta_{i, j_1, j_2, \ldots, j_p} v(x) \geq 0 \), for all \( i, j_1, j_2, \ldots, j_p \neq i \), and \( 1 \leq p \leq m - 1 \), and

A4: \( \Delta_{e_i} v(x) \geq -c. \)

Property A1 implies that \( \Delta_i v(x) \) is non-decreasing in \( x_i \), or equivalently that \( v(x) \) is convex in each of the state variables \( x_i \). Property A2 implies that \( \Delta_i v(x) \) is non-increasing in \( x_j \), or equivalently \( v(x) \) is submodular in the direction of \( (e_i, e_j) \). Property A3 implies that \( \Delta_i v(x) \) is non-decreasing with joint increases in \( x_i, x_{j_1}, \ldots, x_{j_p} \), or equivalently \( v(x) \) is supermodular in the direction of \( (e_i + e_{j_1} + \cdots + e_{j_p}, e_j) \). We are now ready to state and prove the following important lemma.

Lemma 1: If \( v \in \mathcal{V} \) then \( Tv \in \mathcal{V} \), where \( Tv(x) = h(x) + \lambda T_0 v(x) + \sum_{k=1}^{m} \mu_k T_k v(x) \).

Proof: We first prove that the operator \( T \) preserves properties A1-A3. Noting that for the operators \( T_k \), \( k = 1, \ldots, m \),

\[ T_i v(x) = \min \{ v(x + e_i), v(x) \} = v(x) + \min \{ \Delta_i v(x), 0 \}, \text{ and} \]

\[ \Delta_{i} T_i v(x) = \Delta_i v(x) + \min \{ \Delta_i v(x + e_i), 0 \} - \min \{ \Delta_i v(x), 0 \}. \]

We need to distinguish two cases: \( i = k \) and \( i \neq k \).
For the case $i = k$, we have $\Delta T_i v(x) = \Delta v(x) + \min \{ \Delta v(x + e_i), 0 \} - \min \{ \Delta v(x), 0 \}$, and since by A1, we have $\Delta v(x + e_i) \geq \Delta v(x)$, we consider three cases.

Case 1: $\Delta v(x + e_i) \geq \Delta v(x) \geq 0$. Hence $\Delta T_i v(x) = \Delta v(x)$ and therefore we have $\Delta_{j_i} T_i v(x) \geq 0$, $\Delta_{j_i} T_i v(x) \leq 0$, for all $j \neq i$, and $\Delta_{i+j_i} T_i v(x) \geq 0$, for all $j_1, j_2, \ldots, j_p \neq i$.

Case 2: $\Delta v(x + e_i) \geq 0 \geq \Delta v(x)$. Hence $\Delta T_i v(x)$ is identical to the null operator, and hence $\Delta_{i+j_i} T_i v(x) \geq 0$, $\Delta_{j_i} T_i v(x) \leq 0$, for all $j \neq i$, and $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) \geq 0$, for all $j_1, j_2, \ldots, j_p \neq i$.

Case 3: $\Delta v(x) \leq \Delta v(x + e_i) \leq 0$. In this case, $\Delta T_i v(x) = \Delta v(x + e_i)$ and $\Delta_{j_i} T_i v(x) \geq 0$, $\Delta_{j_i} T_i v(x) \leq 0$, for all $j \neq i$, and $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) \geq 0$, for all $j_1, j_2, \ldots, j_p \neq i$.

For the case $i \neq k$, we have (by A2) $\Delta v(x) \geq \Delta v(x + e_i)$. Hence, we also consider three cases.

Case 1: $\Delta v(x) \geq \Delta v(x + e_i) \geq 0$. Hence $\Delta T_i v(x) = \Delta v(x)$ and therefore $\Delta_{j_i} T_i v(x) \geq 0$, $\Delta_{j_i} T_i v(x) \leq 0$, for all $j \neq i$, and $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) \geq 0$, for all $j_1, j_2, \ldots, j_p \neq i$.

Case 2: $\Delta v(x) \geq 0 \geq \Delta v(x + e_i)$. In this case, $\Delta T_i v(x) = \Delta v(x) + \Delta v(x + e_i) = \Delta v(x)$. Hence, by A3, we have $\Delta_{i+j_i} T_i v(x) = \Delta_{i+j_i} v(x) \geq 0$ and therefore $\Delta_{i+j_i} T_i v(x) \geq 0$. Using A2, note that $\Delta_{i+j_i} T_i v(x) = \Delta_{i+j_i} v(x) = \Delta v(x + e_i + e_i) - \Delta v(x) \leq 0$, hence $\Delta_{i+j_i} T_i v(x) \leq 0$, for all $j \neq i$. To prove that $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) \geq 0$ for all $j_1, j_2, \ldots, j_p \neq i$, note that

$$
\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \min \{ v(x + e_i + e_i + \cdots + e_i), v(x + e_i + e_i + \cdots + e_i) \} - \min \{ v(x + e_i), v(x) \} \\
= \Delta_{i+j_i+j_2+\cdots+j_p} v(x) + \min \{ \Delta_1 v(x + e_i + e_i + \cdots + e_i), 0 \} - \min \{ \Delta_1 v(x), 0 \},
$$

Since we have $\Delta_1 v(x) \geq 0$, then $\min \{ \Delta_1 v(x), 0 \} = 0$. Here, we distinguish two situations: either $j_1, j_2, \ldots, j_p \neq k$ or $j_1, j_2, \ldots, j_p = k$. If $j_1, j_2, \ldots, j_p \neq k$, then by A2 we have $\Delta v(x + e_i) \geq \Delta v(x + e_i + e_i + \cdots + e_i)$ and

$$
\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x) + \Delta v(x + e_i + e_i + \cdots + e_i) - \Delta_{i+j_i+j_2+\cdots+j_p} v(x).
$$

Consequently,

$$
\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x) \geq 0 \text{ (by A3)}.
$$

If $j_1, j_2, \ldots, j_p = k$, we consider two cases: $\Delta v(x + e_i + e_i + \cdots + e_i) \geq 0$ and $\Delta v(x + e_i + e_i + \cdots + e_i) < 0$. When $\Delta v(x + e_i + e_i + \cdots + e_i) \geq 0$, we have $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x)$. Therefore, $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x) \geq 0$ (by A3). When $\Delta v(x + e_i + e_i + \cdots + e_i) < 0$, we have $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x)$. Therefore, $\Delta_{i+j_i+j_2+\cdots+j_p} T_i v(x) = \Delta_{i+j_i+j_2+\cdots+j_p} v(x) \geq 0$ (by A3). Hence, $\Delta_{i+j_i+j_2+\cdots+j_p} v(x) \geq 0$, for all $j_1, j_2, \ldots, j_p \neq i$. 

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Case 3: \( \Delta_k v(x + e_i) \leq \Delta_k v(x) \leq 0 \). In this case, \( \Delta_j T_k v(x) = \Delta_j v(x + e_i) \). Hence, \( \Delta_j T_k v(x) \geq 0 \), \( \Delta_j T_k v(x) \leq 0 \), for all \( j \neq i \), and \( \Delta_{i_1 j_1 + \cdots + j_p} T_k v(x) \geq 0 \), for all \( j_1, j_2, \cdots, j_p \neq i \).

A consequence of the above is that for all values of \( k \), we have \( \Delta_j T_k v(x) \geq 0 \), for all \( i \), \( \Delta_j T_k v(x) \leq 0 \), for all \( i \) and \( j \neq i \), and \( \Delta_{i_1 j_1 + \cdots + j_p} T_k v(x) \geq 0 \), for all \( i, j_1, j_2, \cdots, j_p \neq i \), and \( 1 \leq p \leq m - 1 \). Hence, \( T_k v(x) \) satisfies A1-A3.

To show that \( T_0 v(x) \) satisfies A1-A3, we distinguish two cases: \( \prod_k x_k = 0 \), which implies that \( \Delta_0 T_0 v(x) = \Delta_0 v(x) \) and \( \prod_k x_k \neq 0 \) which implies that \( \Delta_0 T_0 v(x) = \Delta_0 v(x - e) \). Hence, \( T_0 v(x) \) satisfies A1-A3.

To show that \( T v \) preserves property A4, note that for the operators \( T_k \) (\( k = 1, \cdots, m \)) we have

\[
\Delta_j T_k v(x) = \min \{ v(x + e_i) + v(x + e_j), v(x + e_j) - v(x) \} - \min \{ v(x + e_i), v(x) \} \]

\[
= \min \{ \Delta_i v(x + e_i), \Delta_j v(x - \Delta_k v(x)) \} + \max \{ \Delta_k v(x), 0 \}.
\]

We distinguish two cases: \( \Delta_k v(x) \geq 0 \) which implies \( \Delta_j T_k v(x) = \min \{ \Delta_i v(x + e_i), \Delta_j v(x) - \Delta_k v(x) \} \geq \min \{ -c + \Delta_j v(x), -c \} = -c \), and \( \Delta_k v(x) < 0 \) which implies \( \Delta_j T_k v(x) = \min \{ \Delta_i v(x + e_i), \Delta_j v(x) - \Delta_k v(x) \} \geq \min \{ -c, -c - \Delta_k v(x) \} = -c \).

Hence \( T_k v(x) \) satisfies A4. For the operator \( T_0 \), we have

\[
\Delta_j T_0 v(x) = T_0 v(x + e) - T_0 v(x) = v(x) - T_0 v(x).
\]

We distinguish two cases: \( \prod_k x_k = 0 \) which implies \( \Delta_j T_0 v(x) = v(x) - v(x - e) = -c \), and \( \prod_k x_k \neq 0 \) which implies \( \Delta_j T_0 v(x) = v(x) - v(x - e) = \Delta_j v(x - e) \geq -c \). Hence \( T_0 v(x) \) satisfies A4.

It is straightforward to show that \( h(x) \) satisfies A1-A3 since \( h(x) \) is separable in the variables \( x_i \) and increasing convex in each \( x_i \). Since \( \overline{v} \) is closed under multiplication by a scalar and addition, \( T v \) satisfies properties A1-A3. To show that \( T v \) also satisfies A4, note that

\[
\Delta_j T v(x) = \Delta_j T h(x) + \lambda \Delta_j T_0 v(x) + \sum_{k=1}^m \mu_k \Delta_j T_k v(x)
\]

\[
\geq \Delta_j T h(x) - \lambda c - \sum_{k=1}^m \mu_k c \quad (\Delta_j T h(x) > 0)
\]

\[
\geq -c \left( \lambda + \sum_{k=1}^m \mu_k \right) = -c(1 - \alpha) \geq -c.
\]

Hence, \( T v \in \overline{v} \), which completes the proof of lemma 1.
Let \( v \in \mathcal{V} \), and define the following quantities.

\[
 s_k(x_{-k}) = \min \{ x_k \geq 0 \mid \Delta_k v(x) > 0 \}.
\]

Then, condition A1 combined with the above definition implies

\[
 \begin{aligned}
 &\Delta_k v(x) > 0 \quad \text{if } x_k \geq s_k(x_{-k}), \\
 &\Delta_k v(x) \leq 0 \quad \text{otherwise}.
\end{aligned}
\]

Conditions A2 and A3 combined with the above definition for \( s_k(x_{-k}) \) imply the following lemma.

**Lemma 2:** Let \( v \in \mathcal{V} \), then

1. \( s_k(x_{-k}) \leq s_k(x_{-k} + e_j) \), and
2. \( s_k(x_{-k} + e_j) \leq s_k(x_{-k}) + 1 \),

for all \( j \neq k \).

**Proof:** From condition A2 and the definition of \( s_k(x_{-k}) \), we have

\[
 \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k} + e_j), x_{k+1}, \ldots, x_m)) \geq \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k}), x_{k+1}, \ldots, x_m) + e_j) \geq 0.
\]

Hence \( \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k}), x_{k+1}, \ldots, x_m)) \geq 0 \). Using the definition of \( s_k(x_{-k}) \), leads to

\[
 s_k(x_{-k}) \leq s_k(x_{-k} + e_j).
\]

From condition A3, we have

\[
 \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k}) + 1, x_{k+1}, \ldots, x_m) + e_j) \geq \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k}), x_{k+1}, \ldots, x_m)) \geq 0.
\]

Hence \( \Delta_k v((x_1, \ldots, x_{k-1}, s_k(x_{-k}) + 1, x_{k+1}, \ldots, x_m) + e_j) \geq 0 \) and using the definition of \( s_k(x_{-k}) \) leads to

\[
 s_k(x_{-k} + e_j) \leq s_k(x_{-k}) + 1,
\]

which completes the proof of lemma 2. \( \blacksquare \)

A result similar to lemma 2 was shown by Gayon et al. (2004) in the context of a make-to-stock queue with advance demand information. In their case, they show that the optimal base-stock level increases by at most one with increases in the number of orders announced but not due yet.

If \( v \) corresponds to the value function of a control policy \( \pi \), then the control policy consists of a base-stock policy with base-stock level \( s_k(x_{-k}) \) such that it is optimal to produce component \( k \) if \( x < s_k(x_{-k}) \) and not to produce otherwise, with the base-stock level having the properties shown in lemma 2. From property A4, the policy also specifies that orders should be fulfilled if on-hand inventory is available.

In order to show that interrupting the production of a component is never optimal, it suffices to notice that decisions are made only at times when the system state changes (an order arrival or a component completion). If it is optimal to produce a component \( k \) in state \( x = (x_1, \bar{O}_1, x_m) \), then it continues to be optimal to produce it if an order arrives which either leaves the state unchanged (in case of unavailability of a component) or moves it to state \( x-e = (x_1-1, \bar{O}_1, x_m-1) \), by virtue of the fact that the optimal policy is a base-stock policy and the base-stock level for component \( k \) is non-decreasing in the variables \( x_j \).
Similarly, if it is optimal to produce component \( k \) in state \( x \), then it continues to be optimal to produce it if a component \( j \) \((j \neq k)\) completes and move the state from \( x \) to \( x + e_k \), since the optimal base-stock level for component \( k \) is again non-decreasing in the variables \( x_j \).

To complete the proof of theorem 1, we need to show that \( v^* \in \mathcal{V} \). This can be easily shown by noting that (1) \( v^* = \lim_{n \to \infty} T^n(v) \) for any \( v \) in \( \mathcal{V} \), where \( T^n \) refers to \( n \) compositions of operator \( T \) and (2) \( v^* \) is the unique solution of \( v^* = T(v) \) (see Theorem 5.1 of Porteus (1982) and Theorem 6.10.4 of Puterman (1994)). Hence, \( v^* \) satisfies conditions A1-A4 and lemma 2, from which the results of theorem 1 immediately follow.

**Proof of Theorem 2**

First, define \( \mathcal{U} \) as the set of functions on \( \mathbb{R}^m \) such that if \( v \in \mathcal{U} \), then for all \( i = 1, \ldots, m \):

**C1:** \( \Delta_j v(y) \geq 0 \), for all \( i \),

**C2:** \( \Delta_{ij} v(y) \leq 0 \), for all \( i \) and \( j \neq i \),

**C3:** \( \Delta_{i_{1}j_{1}i_{2}j_{2} \cdots j_{p}} v(y) \geq 0 \), for all \( i, j_1, j_2, \ldots, j_p \neq i \), and \( 1 \leq p \leq m - 1 \), and

**C4:** \( \Delta_{ij} v(y) \leq 0 \), for \( y_j < 0 \).

**Lemma 3:** If \( v \in \mathcal{U} \), then \( T(v) \in \mathcal{U} \).

**Proof:** The proof for showing that \( T_k v \in \mathcal{U} \) for \( k = 1, \ldots, m \) is the same as in lemma 1. To show that \( z(y) \) satisfies condition C1-C3, notice that \( \Delta_{i} B(y) = B(y + e_i) - B(y) = -b \) or 0, for any vector \( y \). Hence, \( \Delta_{ij} B(y) = 0 \), \( \Delta_{i} B(y) = 0 \), and \( \Delta_{i_j} B(y) = 0 \). Therefore, \( B(y) \) satisfies conditions C1-C3. To prove C4, assume \( y < 0 \). Then, if \( i = k \), we have \( \Delta_{i} T_v(y) = \min \{ \Delta_{i} v(y + e_i), 0 \} \leq 0 \) (since \( \Delta_{i} v(y) \leq 0 \) by C4). If \( i \neq k \), we have \( 0 \geq \Delta_{i} v(y) \geq \Delta_{i} v(y + e_i) \) (by C4 and C2) which implies that \( \Delta_{i} T_v(y) = \Delta_{i} v(y + e_i) \leq \Delta_{i} v(y) \leq 0 \). We also have \( \Delta_{i} T_0 v(y) = \Delta_{i} v(y - e) - \Delta_{i} v(y) = -\Delta_{i} v(y - e) \leq 0 \) (by C3) and \( \Delta_{i} B(y) \leq 0 \). Since \( \mathcal{U} \) is closed under addition and multiplication by a scalar, the result follows immediately. ■

The rest of the proof is similar to that of theorem 1. For brevity, the details are omitted.
Proof of Theorem 3

Let \( \bar{w} \) be the set of functions on \( \Box^m \) such that if \( v \in \bar{w} \), then:

D1: \( \Delta_{ij}v(x) \geq 0 \), for all \( i \),

D2: \( \Delta_{ij}v(x) \leq 0 \), for all \( i \) and \( j \neq i \),

D3: \( \Delta_{1+1+\cdots+1}v(x) \geq 0 \), for all \( i, j_1, j_2, \ldots, j_p \neq i \), and \( 1 \leq p \leq m-1 \), and

D4 \( \Delta_xv(x) \geq -c_1 \).

Lemma 4: If \( v \in \bar{w} \), then \( T_v \in \bar{w} \).

Proof: In Lemma 1, we showed that \( T_k v \) satisfies properties D1 through D3 (which are the same as A1-A3). We now show that \( T_k v \) also satisfies D4. First note that \( \Delta_x T_k v(x) \) can be written as

\[
\Delta_x T_k v(x) = \min \{ \Delta_x v(x + e_1), \Delta_x v(x) - \Delta_x v(x) \} + \max \{ \Delta_x v(x), 0 \}.
\]

We distinguish two cases.

1. \( \Delta_x v(x) \geq 0 \) implies \( \Delta_x T_k v(x) = \min \{ \Delta_x v(x + e_1), \Delta_x v(x), \Delta_x v(x) \} = \Delta_x v(x) \).

2. \( \Delta_x v(x) < 0 \) implies

\[
\Delta_x T_k v(x) = \min \{ \Delta_x v(x + e_1), \Delta_x v(x) - \Delta_x v(x) \} \geq \min \{ -c_1, -c_1 - \Delta_x v(x) \} = -c_1.
\]

Hence, \( T_v \) satisfies D4. Hence, \( T_v \in \bar{w}, \ k = 1, \ldots, m \).

Next, we show that \( T_0 v(x) \in \bar{w} \). First, note that

\[
\Delta_x T_0 v(x) = \begin{cases} 
  \Delta_x v(x) + c_i & \text{if } \prod_{j \neq i} x_j = 0 \\
  \min \{ v(x + e_j) - v(x), v(x + e_i) + c_i \} & \text{otherwise}
\end{cases}
\]

\[
- \begin{cases} 
  \Delta_x v(x) + c_i & \text{if } \prod_{j \neq i} x_j = 0 \\
  \min \{ v(x - e_j), v(x) \} & \text{otherwise}.
\end{cases}
\]

If \( \prod_{j \neq i} x_j = 0 \), then \( \Delta_x T_0 v(x) = \Delta_x v(x) \). If \( \prod_{j \neq i} x_j = 0 \) and \( x_i = 0 \), then

\[
\Delta_x T_0 v(x) = \min \{ v(x + e_j - e), v(x + e_j) + c_i \} - v(x) - c_i
\]

\[
= \min \{ v(x + e_j - e) - v(x) - c_i, \Delta_x v(x) \}
\]

\[
= \begin{cases} 
  \Delta_x v(x) & \text{if } \Delta_x v(x + e_j - e) < -c_i \\
  v(x + e_j - e) - v(x) - c_i & \text{if } \Delta_x v(x + e_j - e) \geq -c_i.
\end{cases}
\]

If, on the other hand, \( \prod_{j \neq i} x_j \neq 0 \) and \( x_i \neq 0 \), then
$$
\Delta T_0'v(x) = \min \{ v(x + e_i - e), v(x + e_i) + c_i \} - \min \{ v(x - e), v(x) + c_i \}
$$
$$
= \min \{ v(x + e_i - e) - v(x) - c_i, \Delta v(x) \} + \max \{ 0, \Delta v(x) + c_i \}
$$
$$
= \begin{cases} 
\Delta v(x) & \text{if } \Delta v(x) < -c_i \\
\Delta v(x - e) & \text{if } \Delta v(x - e) \geq -c_i
\end{cases}
$$

To summarize, we have

\[
\Delta T_0'v(x) = \begin{cases} 
\Delta v(x) & \text{if } \prod x_j = 0 \\
\Delta v(x) & \text{if } \prod (x_j \neq 0, x_j = 0, \text{and } \Delta v(x - e) < -c_i) \\
v(x + e_i - e) - v(x) - c_i & \text{if } \prod x_j \neq 0, x_j = 0, \text{and } \Delta v(x + e_i - e) \geq -c_i \\
\Delta v(x) & \text{if } \prod x_j \neq 0, x_j \neq 0, \text{and } \Delta v(x - e) < -c_i \\
\Delta v(x - e) & \text{if } \prod x_j \neq 0, x_j \neq 0, \text{and } \Delta v(x - e) \geq -c_i
\end{cases}
\]

Therefore, it is sufficient to show that \( v(x + e_i - e) - v(x) - c_i \) satisfies D1-D4. First note that
\[
\Delta T_0'v(x) = v(x + e_i - e) - v(x) - c_i = \Delta_j v(x - e) - \Delta v(x - e) - c_i.
\]

**Condition D1**

First, note that
\[
\Delta_{i+j_1+\ldots+j_p} v(x) = \Delta_j v(x + e_i + e_{j_1} + \ldots + e_{j_p}) - \Delta v(x)
\]
$$
\leq \Delta_j v(x + e_i + e_{j_1} + \ldots + e_{j_p}) - \Delta v(x) = \Delta_{i+j_1+\ldots+j_p} \Delta v(x)
$$
$$
\vdots
$$
$$
\leq \Delta v(x + e_i) - \Delta v(x) = \Delta_{i} \Delta v(x).
$$

A consequence of the above is that \( \Delta_{i} \Delta v(x - e) \geq \Delta_{i} \Delta v(x - e) \). Now since
\[
\Delta \Delta T_0'v(x) = \Delta_{i} \Delta v(x - e) - \Delta_{i} \Delta v(x - e),
\]
we have \( \Delta_{i} \Delta T_0'v(x) \geq 0 \).

**Condition D2**

\( \Delta_j \Delta v(x - e) \leq 0 \) and \( \Delta_j \Delta v(x - e) \geq 0 \). Hence, \( \Delta_j \Delta T_0'v(x) \leq 0 \).
Condition D3

\[
\Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = \begin{cases} 
\{v(x + e_i + e_{j_i} + \cdots + e_{j_j}) + c_i \} & \text{if } \prod_{j \neq j_i, \ldots, j_j} x_j = 0 \\
\min \{v(x + e_i + e_{j_i} + \cdots + e_{j_j} - e_j), v(x + e_i + e_{j_i} + \cdots + e_{j_j}) + c_i \} & \text{otherwise}
\end{cases}
\]

Since \( \prod_{j \neq j_i, \ldots, j_j} x_j \neq 0 \) and \( x_j = 0 \), we have

\[
\Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = \min \{v(x + e_i + e_{j_i} + \cdots + e_{j_j} - e), v(x + e_i + e_{j_i} + \cdots + e_{j_j}) + c_i \} - v(x) - c_i
\]

If \( \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = v(x - e) - \Delta_{i, j} v(x - e) - c_i > 0 \), then \( \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) \), which implies

\[
\Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) \geq 0.
\]

If \( \Delta_{i, j, k, \ldots, l}^{\text{ij} j} v(x) = v(x - e) - \Delta_{i, j} v(x - e) - c_i \), then,

\[
\Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) - \Delta_{i, j} v(x - e) - c_i.
\]

Therefore, \( \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) = \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) - \Delta_{i, j} v(x - e) \). Since

\[
\Delta_{i, j} v(x) = \Delta_{i} v(x + e) - \Delta_{i} v(x)
\]

we have \( \Delta_{i, j, k, \ldots, l}^{\text{ij} j} T_0^i v(x) \geq 0. \)

Condition D4

To verify D4, first notice that

\[
\Delta_{i} T_0^{ij} v(x) = \min \{v(x), v(x + c)\} = \begin{cases} 
v(x) + c_i & \text{if } \prod_{j} x_j = 0 \\
\min \{v(x - e), v(x) + c_i \} & \text{otherwise}
\end{cases}
\]

Since \( x_j = 0 \), we have
\[ \Delta T_0^i v(x) = \min \left\{ v(x), v(x + e) + c_j \right\} - v(x) - c_i \]
\[ = \min \left\{ -c_i, \Delta_x v(x) \right\} \geq \min \left\{ -c_i, -c_i \right\} = -c_i. \]

Hence, \( T_0^i v(x) \) satisfies Condition D4.

Finally, since the function \( h(x) \) is separable in the variables \( x_i \) and increasing convex in each \( x_i \), then \( h(x) \in \overline{\omega} \). Since \( \overline{\omega} \) is closed under addition and multiplication by a scalar, we have \( Tv \in \overline{\omega} \). \( \blacksquare \)

Let \( v \in \overline{\omega} \), and define the following quantities.
\[ s_k(x_{-k}) = \min \left\{ x_k \geq 0 \mid \Delta_x v(x) > 0 \right\}, \text{ and} \]
\[ r_{k,j}(x_{-j}) = \min \left\{ x_k \geq 0 \mid \Delta_x v(x - e) > -c_j \right\}. \]

Then, condition D1 combined with the above definition implies
\[ \begin{cases} 
\Delta_x v(x) > 0 & \text{if } x_k \geq s_k(x_{-k}) , \\
\Delta_x v(x) \leq 0 & \text{otherwise.} \end{cases} \]

Conditions D2 and D3 combined with the above definition for \( s_k(x_{-k}) \) imply and
\[ s_k(x_{-k} + e_j) \leq s_k(x_{-k}) + 1 \text{ for all } j \neq k. \]

Similarly, using the fact that \( \Delta_x v(x) \geq 0, \) (by virtue of D3) and the above definition for \( r_{k,j}(x_{-j}) \) we have
\[ \begin{cases} 
\Delta_x v(x - e) \geq -c_i & \text{if } x_k \geq r_{k,j}(x_{-j}), \\
\Delta_x v(x - e) \leq -c_i & \text{otherwise.} \end{cases} \]

Condition D4 specifies that it is always optimal to satisfy orders from class 1.

If \( v \) corresponds to the value function of a control policy \( \pi \), then the control policy consists of a base-stock policy with base-stock level \( s_k(x_{-k}) \) such that it is optimal to produce component \( k \) if \( x_k < s_k(x_{-k}) \) and not to produce otherwise, with the base-stock level having the properties shown in lemma 2. The policy also specifies rationing levels for each demand class such that it is optimal to satisfy an order from class \( l \) only if \( x_k \geq r_{k,l}(x_{-k}) \) for \( k = 1, \ldots, m \). The rationing levels are ordered \( r_{k,\nu}^*(x_{-k}) \geq \ldots \geq r_{k,1}^*(x_{-k}) = 0 \) since \( \Delta_x v(x - e) \) is increasing in \( x_i \) and the backorder costs are ordered as \( c_1 \geq \ldots \geq c_n \). The equality \( r_{k,l}^*(x_{-k}) = 0 \) follows from condition 4 and means that it is always optimal to satisfy orders from class 1.

To complete the proof, we must show that \( v^* \in \overline{\omega} \), which can be done per the proof of theorem 1.